Turbulent Transport Equations and Kubo Formulae for Eddy Transport Coefficients

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Kubo type time correlation formulae for turbulent transport coefficients in incompressible but heat conducting fluids are derived, especially for eddy viscosity, eddy heat conductivity, and pressure. The connection to the cascade method as well as its equivalence to the methods of closure of hierarchy are established. Lagrangean time integration is used. If the retarded Green's function has exponential time behaviour the damping constant Γ_q can be calculated explicitely. In this approximation in the inertial range one finds the Kolmogoroff $k^{-5/3}$ spectrum including its numerical factor ($C=1.53$). This induces a frequency spectrum $\sim \omega^{-2}$.

1. Introduction

There are two main problems in turbulent flow of fluids. One wants to describe the fluctuations, i. e. the irregular eddies, around a certain mean flow pattern and one is interested in the time development of a non-equilibrium fluid. Both aspects are closely connected. Nevertheless one is tempted to separate them as much as possible in order to simplify the theory. Transport in turbulence has therefore been treated by simple models connected with the concepts "mixing length", "eddy viscosity", "cascade" (for a review see e. g. [1]). At the other hand, homogeneous and isotropic stationary turbulence has been treated by some closure of the hydrodynamic hierarchy, e. g. by KRAICHNAN [2] in a series of papers (for a review see [3]), WYLD [4], ORSZAG (for a review of these "analytical" methods see the excellent paper by ORSZAG [5]), HERRING [6], BALESCU and SENATORSKI [7], and others. As a connection between these treatments one should mention e. g. HEISENBERG's theory [8] and similar ones. All theories are more or less based on the fundamental picture of turbulent motion derived by KOLMOGOROFF [9], which is excellently described by LANDAU and LIFSCHITZ [10].

Recently, the cascade technique has been considerably improved by using Lagrangean time integration for the fluctuations by TCHEN [11]. Now, this method seems to be very useful for spectral problems as well as for applications to non-equilibrium problems: Electrostatic fluctuations in turbulent plasmas [12], turbulence with magnetic field [13]; ionospheric two component plasmas coupled to the neutral back ground [14], influence of turbulence on chemical reactions [14]. Turbulence effects change the macroscopic transport equations. The additional terms are characterized by eddy coefficients, represented by some kind of time integral formula.

The aim of this paper is to extend the cascade technique, to derive Kubo transport formulae for the eddy coefficients, to calculate the turbulent spectrum, and to show the connection to hierarchy closure. The cascade method used by Tchen divides the eddies into two groups, interacting with each other.

$$\boldsymbol{u}^0(\boldsymbol{x}, t \mid k) = \int_0^k \mathrm{d}^3 p \exp\{i \boldsymbol{p} \cdot \boldsymbol{x}\} \boldsymbol{u}(\boldsymbol{p}, t), \quad (1\,\mathrm{a})$$

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[2] R. H. KRAICHNAN, J. Fluid Mech. 5, 497 [1959]; **9**, 1728 [1966].
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[11] C. M. TCHEN, Cascade Mechanism in the Wave Mixing Processes of Turbulence, in: Nonlinear Effects in Plasmas, Ed. G. KALMAN and M. FEIX, Gordon and Breach 1969.
[12] C. M. TCHEN, in: Lectures in Theoretical Physics, Part C, 1967, Ed. BRITTIN et al.
[13] C. M. TCHEN, IEEE Transactions, Vol. ED-17, 247 [1970].
[14] C. M. TCHEN, Preprint, 1971.

$$\boldsymbol{u}'(\boldsymbol{x}, t \mid k) = \int_k^\infty \mathrm{d}^3 p \exp\{i \boldsymbol{p} \cdot \boldsymbol{x}\} \boldsymbol{u}(\boldsymbol{p}, t). \quad (1\text{ b})$$

Instead, in the following all modes shall be treated individually. The smoothing procedure of different scales in space according to Eqs. (1) is replaced by an average, defined by a non-equilibrium initial ensemble.

The macroscopic flow will be governed by the net effect of the small scale fluctuations in form of an (nearly) equilibrium average of correlations at different times. Eddy viscosity or eddy thermal conductivity are given as k, $\omega \to 0$ limits of correlation functions. This general picture is very familiar in transport theory now (see e. g. [15, 16] for reviews), and denoted as Kubo formulae. The time development used here is of Lagrangean type.

In lowest approximation the cascade results are recovered. More general treatment describes turbulent flow as a non Newtonian fluid with higher order transport effects. Especially second order transport effects can be related to moments of the time correlation function, which characterizes the eddy viscosity. In general the fluid has a memory of the order of the decay time of the fluctuations.

If the relaxation time approximation is applied to the general formula, the Kolmogoroff-Obukhoff-law $F(k) = C\,\varepsilon_{\mathrm{dis}}^{2/3} k^{-5/3}$ is obtained for the inertial subrange; $C = 1.53$. The approach to the viscous range obeys a k^{-7}-law; the coefficient is 3 times larger than in Heisenberg's theory. The half width of the turbulent frequency spectrum, Γ_k, is also given. It is $\sim k^{2/3}$ in the inertial subrange and approaches $k^2 \nu$ as $\sim k^{-6}$ for large wave numbers, see Eq. (37). By scaling one finds the time correlation spectrum $\varphi_{uu}(\omega) \sim \omega^{-2}$ in the inertial and $\sim \omega^{-4}$ in the viscous range.

At the other hand, if one avoids the eddy-viscosity approximation, the method presented here can also be applied to isotropic, homogeneous and stationary turbulence. Then one easily derives the equations of Kraichnan's direct interaction approximation but *including* the Lagrangean time development in the Green's function. Thus the connection between cascade methods and hierarchy closure is established as well as the extension to non-equilibrium situations.

In order to show applications beyond the velocity turbulence we treat an incompressible fluid coupled to a variable temperature field $T(\boldsymbol{x}, t)$. The eddy thermal conductivity is essentially given by the eddy viscosity, if the molecular friction and heat conduction are small, i. e. especially in the inertial range. In the latter *both* the velocity spectrum and the temperature spectrum are $\sim k^{-5/3}$. This has also been found by TCHEN [11, 12].

2. Equations of Non-Equilibrium Motion

The general equations of motion of an incompressible, heat conducting fluid are [10]:

$$\operatorname{div} \boldsymbol{u} = 0, \quad (2\text{ a})$$

$$(\partial_t + \boldsymbol{u} \cdot \nabla)\, \boldsymbol{u} = -\nabla P + \nu \nabla^2 \boldsymbol{u}, \quad (2\text{ b})$$

$$(\partial_t + \boldsymbol{u} \cdot \nabla)\, T = \chi \nabla^2 T. \quad (2\text{ c})$$

These 5 equations for the fields $\boldsymbol{u} \equiv (u_1, u_2, u_3) \equiv (u_i)$, T, and P (the pressure, divided by the constant density) determine the mean motion as well as the fluctuations.

The fluctuations are governed by the turbulent initial conditions, more specifically by the phases of the turbulent modes, which give rise to the statistical degrees of freedom (see [10]). In fully developed turbulence these phases can be treated as statistically distributed. Therefore the fields $\boldsymbol{u}$, T, P are random variables in so far, as they are influenced by the turbulent phases. At the same time they are of none random nature, in as much as macroscopic initial conditions and boundary values are concerned.

To express this quantitatively, we use wave vector representation. Each field is transformed as usual by

$$u_i(\boldsymbol{x}, t) = \sum_k \exp\{i\, \boldsymbol{k} \cdot \boldsymbol{x}\}\, u_i(\boldsymbol{k}, t), \; u_i^*(\boldsymbol{k}) = u_i(-\boldsymbol{k}),$$

$$u_i(\boldsymbol{k}, t) = \int \frac{\mathrm{d}^3 x}{V} \exp\{-i\, \boldsymbol{k} \cdot \boldsymbol{x}\}\, u_i(\boldsymbol{x}, t),$$

$$V = L^3, \quad \boldsymbol{k} = \frac{2\pi}{L} \boldsymbol{n}, \boldsymbol{n} = \{n_i\}, \quad n_i = 0, \pm 1, \ldots$$

The equations of motion then read

$$k_j\, u_j(\boldsymbol{k}, t) = 0, \quad (3\text{ a})$$

[15] R. ZWANZIG, Ann. Rev. Phys. Chem. **16**, 67 [1965].

[16] S. GROSSMANN, Wärme und Stoffübertragung **3**, 19 [1970].

$$\partial_t u_i(\boldsymbol{k}, t) + i k_j \sum_q u_j(\boldsymbol{k}-\boldsymbol{q}, t)\, u_i(\boldsymbol{q}, t) = -i k_i P(\boldsymbol{k}, t) - \nu k^2 u_i(\boldsymbol{k}, t), \quad (3\,\mathrm{b})$$

$$\partial_t T(\boldsymbol{k}, t) + i k_j \sum_q u_j(\boldsymbol{k}-\boldsymbol{q}, t)\, T(\boldsymbol{q}, t) = -\chi k^2 T(\boldsymbol{k}, t). \quad (3\,\mathrm{c})$$

The continuity Eq. (3 a), here especially the incompressibility conditon means that the velocity vector $\boldsymbol{u}(\boldsymbol{k})$ has no longitudinal component. Therefore the projector on the transversal part is introduced, $P_{li}(\boldsymbol{k}^0) = \delta_{li} - k_l^0 k_i^0$. The information $P_{li}^0(k^0)\, u_i(k) = u_l(k)$ exhausts (3 a).

The velocity field is determined by the transversal part of Eq. (3 b).

$$\partial_t u_l(\boldsymbol{k}, t) + \frac{i}{2} P_{lij}(\boldsymbol{k}) \sum_q u_j(\boldsymbol{k}-\boldsymbol{q}, t)\, u_i(\boldsymbol{q}, t) = -\nu k^2 u_l(\boldsymbol{k}, t). \quad (4)$$

It is

$$P_{lij}(\boldsymbol{k}) = P_{lji}(\boldsymbol{k}) := P_{li}(k^0)\, k_j + P_{lj}(k^0)\, k_i\,.$$

Once u_l is known, the longitudinal part of Eq. (3 b) determines the pressure,

$$P(\boldsymbol{k}, t) = -k_i^0 k_j^0 \sum_q u_j(\boldsymbol{k}-\boldsymbol{q}, t)\, u_i(\boldsymbol{q}, t), \quad (5)$$

and the temperature is the solution of the linear Eq. (3 c).

A typical picture of the velocity field is shown in Fig. 1. For small k the amplitudes of the modes are determined by the macroscopic conditions. They do

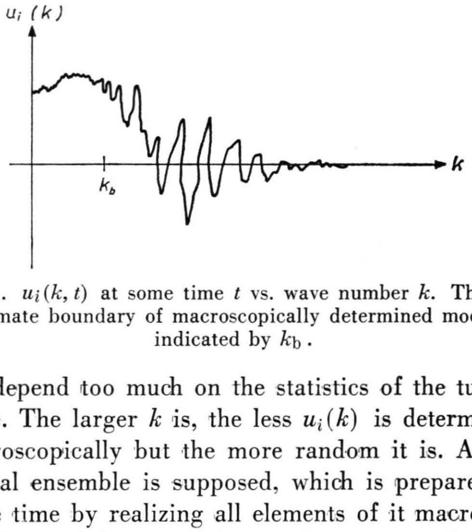

Fig. 1. $u_i(k, t)$ at some time t vs. wave number k. The approximate boundary of macroscopically determined modes is indicated by k_b.

not depend too much on the statistics of the turbulence. The larger k is, the less $u_i(k)$ is determined macroscopically but the more random it is. A statistical ensemble is supposed, which is prepared at some time by realizing all elements of it macroscopically equal but at random with respect to the turbulent degrees of freedom [10]. Ensemble averages $\langle u_i \rangle \equiv \bar{u}_i$ then describe the macroscopic (in general non-equilibrium) behaviour, see Figure 2. The fluctuations $u_i' = u_i - \bar{u}_i$ are the difference between the actual value and the mean value. They represent the random element in the fields, see Figure 3.

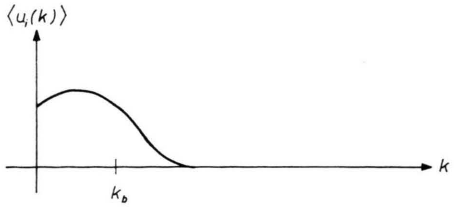

Fig. 2. The ensemble average $\langle u_i(k, t) \rangle$ vs. wave number k.

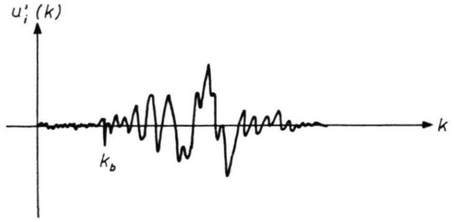

Fig. 3. The fluctuation $u - \langle u \rangle = u'$ vs. wave number k.

The mean quantities depend on the time as expectation values in the Heisenberg picture. I. e. the ensemble is time independent, and the observables $\boldsymbol{u}$, P, T vary in time via equations of motion (3).

If after some initial time $\bar{u}_i$ is independent of t (e. g. especially zero), we have the special case of stationary turbulence. The more non-equilibrium features the ensemble has, the more k-values are expected to be involved in $\bar{u}_i(k, t)$ and the longer the decay to final equilibrium lasts.

Remark: The cascade technique idealizes $\bar{u}$ and thus u' of Figs. 2, 3 by step functions. The boundary, $k = k_\mathrm{b}$, is treated as the cascade separating variable, i. e. $\bar{u}$ is defined by suming in Eq. (1) up to k_b.

The macroscopic equations of motion are derived by averaging Eq.'s (3).

As $\langle \ldots \rangle$ and ∂_t, $\partial_j \ldots$ commute, we find

$$\partial_t \bar{u}_l(\boldsymbol{k}) + \tfrac{1}{2} i P_{lij}(\boldsymbol{k}) \sum_q \bar{u}_j(\boldsymbol{k}-\boldsymbol{q})\, \bar{u}_i(\boldsymbol{q}) = \frac{-i}{2} P_{lij}(\boldsymbol{k}) \sum_q \langle u_j'(\boldsymbol{k}-\boldsymbol{q})\, u_i'(\boldsymbol{q}) \rangle - \nu k^2 \bar{u}_l(\boldsymbol{k}), \quad (6\,\mathrm{a})$$

$$\partial_t T(\boldsymbol{k}) + i k_j \sum_q \bar{u}_j(\boldsymbol{k}-\boldsymbol{q})\, T(\boldsymbol{q}) = -i k_j \sum_q \langle u_j'(\boldsymbol{k}-\boldsymbol{q})\, T'(\boldsymbol{q}) \rangle - \chi k^2 T(\boldsymbol{k}), \quad (6\,\mathrm{b})$$

$$P(\boldsymbol{k}) = -k_i^0 k_j^0 \sum_q \bar{u}_j(\boldsymbol{k}-\boldsymbol{q})\, \bar{u}_i(\boldsymbol{q}) - k_i^0 k_j^0 \sum_q \langle u_j'(\boldsymbol{k}-\boldsymbol{q})\, u_i'(\boldsymbol{q}) \rangle. \quad (6\,\mathrm{c})$$

These are the non-equilibrium equations of mean motion, if the fluctuating forces can be expressed by mean values via closure. This type of equations is known since Reynolds; a closure will be discussed in Section 3.

We need $\langle \ldots \rangle$ for small k, in the macroscopic wave vector regime. Here generally *no* wave number conservation holds and *no* isotropy argument is valid for $\langle \ldots \rangle$, as we have a non-equilibrium ensemble.

3. Equations of Motion for the Fluctuations

These are found by subtracting the mean Eq.'s (6) from the microscopic equations. At first we deal with the velocity fluctuations as the origin of all turbulent features.

$$\partial_t u_l'(\boldsymbol{k}) + \frac{i}{2} P_{lij}(\boldsymbol{k}) \sum_q \{u_j'(\boldsymbol{k}-\boldsymbol{q})\, \bar{u}_i(\boldsymbol{q}) + \bar{u}_j(\boldsymbol{k}-\boldsymbol{q})\, u_i'(\boldsymbol{q}) + u_j'(\boldsymbol{k}-\boldsymbol{q})\, u_i'(\boldsymbol{q}) - \langle u_j'(\boldsymbol{k}-\boldsymbol{q})\, u_i'(\boldsymbol{q})\rangle\} = -\nu k^2 u_l'(\boldsymbol{k}).$$

We now introduce an important assumption. The equation for the fluctuating velocity u_l' contains a mean value term. As u_l' is expected to vary quickly with k, t, and as only this quickly varying part is needed to calculate the turbulent shear, we neglect the average term, being nearly a constant. Then

$$\partial_t u_l'(\boldsymbol{k}) + \frac{i}{2} P_{lij}(\boldsymbol{k}) \sum_q u_j(\boldsymbol{k}-\boldsymbol{q})\, u_i'(\boldsymbol{q}) + \nu k^2 u_l'(\boldsymbol{k}) = -\frac{i}{2} P_{lij}(\boldsymbol{k}) \sum_q \bar{u}_i(\boldsymbol{q})\, u_j'(\boldsymbol{k}-\boldsymbol{q}). \tag{7}$$

The l.h.s. contains the *fully nonlinear* operator, acting on the fluctuation u_l' only instead of u_l itself. On the r.h.s. there is a driving force $\sim k_j \bar{u}_i$, which in position space is $\sim \partial \bar{u}_i / \partial x_j$. This is the origin for an eddy shear stress in the macroscopic equations (6).

The idea now is (TCHEN[11]): (7) can formally be integrated. The differential equation then becomes an integral equation. This integral representation is used to calculate the turbulent shear forces in Eq. (6).

The formal solution of Eq. (7) evidently contains the Lagrangean varying mode $\int dt' \ldots u_j'(t' \mid k-q, t)$ on its r.h.s. To make things clear, we introduce matrix notation

$$\sum_q \left[\delta(\boldsymbol{k},\boldsymbol{q})\, \delta_{li}(\partial_t + \nu k^2) + \frac{i}{2} P_{lij}(\boldsymbol{k})\, u_j(\boldsymbol{k}-\boldsymbol{q},t)\right] u_i'(\boldsymbol{q},t) = -\frac{i}{2} P_{lmn}(\boldsymbol{k}) \sum_q \bar{u}_m(\boldsymbol{k}-\boldsymbol{p},t)\, u_n'(\boldsymbol{p},t).$$

The matrix

$$G_{li}^{-1}(k\,t, q\,t') \equiv \delta(t,t') \cdot \left[\delta_{li}\, \delta(\boldsymbol{k},\boldsymbol{q})\, (\partial_{t'} + \nu k^2) + \frac{i}{2} P_{lij}(\boldsymbol{k})\, u_j(\boldsymbol{k}-\boldsymbol{q},t')\right]$$

describes the Lagrangean motion, molecular damping included. It is not exactly the inverse nonlinear Green's function for the velocity field, but similar. The inverse matrix G is an integral operator, which depends on $\boldsymbol{u}$. The formal solution reads

$$u_i'(\boldsymbol{q},t) = \frac{-i}{2} \int dt' \sum_{q'} \sum_p G_{il}(\boldsymbol{q}, \mathrm{t} \mid \boldsymbol{q}'\, t')\, P_{lmn}(\boldsymbol{q}') \cdot \bar{u}_m(\boldsymbol{p},t')\, u_n'(\boldsymbol{q}'-\boldsymbol{p},t'). \tag{8}$$

From physical reasons we need the causal nonlinear Green's function. Therefore $t' \leqq t$ limits the upper integration.

In order to see the physical meaning of $G_{ll'}(q\,t \mid q'\,t')$, we compare it with the exact, nonlinear Green's function $\delta u_l(q\,t \mid f) / \delta f_{l'}(q', t')$. This can be found from the equation of motion of the velocity field under external forces f.

Functional derivative with respect to $f_{l'}(k', t')$ and then $f \to 0$ gives the response equation.

$$\sum_q \{(\partial_t + \nu k^2)\, \delta(\boldsymbol{k},\boldsymbol{q})\, \delta_{ln} + i P_{lmn}(\boldsymbol{k})\, u_m(\boldsymbol{k}-\boldsymbol{q},t)\} \cdot \frac{\delta u_n(\boldsymbol{q},t)}{\delta f_{l'}(\boldsymbol{k}',t')} = \delta_{ll'}\, \delta(\boldsymbol{k},\boldsymbol{k}')\, \delta(t,t').$$

Because of the nonlinearity, instead of $\frac{1}{2} P_{lmn}$ as in (7) we find twice this. Therefore G differs from the response function but has a similar interpretation as "response" at time t, mode q, component l, due to a disturbance at time t' in mode q' with component l'. G will decay for large $t-t'$ and $q-q'$.

The formal solution (8) now can be used to calculate the turbulent shear in the equation of mean motion.

$$\sum_q \langle u_j'(\boldsymbol{k}-\boldsymbol{q},t)\, u_i'(\boldsymbol{q})\rangle = -\frac{i}{2} \int dt' \sum_{\substack{qq'\\p}} P_{rmn}(\boldsymbol{q}')\, \bar{u}_m(\boldsymbol{p},t') \cdot \langle u_j'(\boldsymbol{k}-\boldsymbol{q},\mathrm{t})\, G_{ir}(\boldsymbol{q}\,\mathrm{t} \mid \boldsymbol{q}'\,t')\, u_n'(\boldsymbol{q}'-\boldsymbol{p},t')\rangle. \tag{9}$$

4. The Transport Coefficient Approximation

Let us discuss first the physical contents of the turbulent shear effects by deriving the transport equation and the time correlation function representation for the relevant transport coefficient. (i) The macroscopic driving term, $\bar{u}_m(\boldsymbol{p}, t')$, will only slowly vary with respect to t', compared to the turbulent time scale, which $G_{ir}(q\,t \,|\, q'\,t')$ governs. This allows $t' \to t$ in $\bar{u}_m$ as lowest order approximation. (ii) The most important contribution to the t' integral is for small $t - t'$. Then $q \approx q'$ in G. (iii) As the turbulent wave vector region is locally homogeneous in the time average, we have momentum conservation in the expectation value:

$$\boldsymbol{k} - \boldsymbol{q} = -(\boldsymbol{q}' - \boldsymbol{p}) \approx -(\boldsymbol{q} - \boldsymbol{p}), \quad \text{i. e.} \quad \boldsymbol{p} \approx \boldsymbol{k}.$$

Therefore in the mean velocity $\bar{u}_m$, $\boldsymbol{p}$ can be approximated by $\boldsymbol{k}$. Using these approximations the turbulent shear effect turns out to be $\sim i\, k_n\, \bar{u}_m(\boldsymbol{k}, t)$, which in position space just is the macroscopic shear rate $\partial \bar{u}_m / \partial x_n$. The factor left can be interpreted as eddy viscosity.

The *final result*, to be used in (6 a) is:

$$-\frac{i}{2}\, P_{lij}(\boldsymbol{k}) \sum_q \langle u_j'(\boldsymbol{k} - \boldsymbol{q})\, u_i'(\boldsymbol{q}) \rangle = -k^2\, \nu_{lm}(k, t)\, \bar{u}_m(\boldsymbol{k}, t) \qquad (10)$$

with the turbulent viscosity matrix

$$\nu_{lm}(k, t) = \frac{1}{4\,k^2} \sum_{q, q', p} P_{lij}(\boldsymbol{k})\, P_{rmn}(\boldsymbol{q}') \int_0^t \mathrm{d}t' \cdot \langle u_j'(\boldsymbol{k} - \boldsymbol{q}, t)\, G_{ir}(\boldsymbol{q}\,t \,|\, \boldsymbol{q}'\,t')\, u_n'(\boldsymbol{q}' - \boldsymbol{p}, t') \rangle. \qquad (11)$$

Discussion of this Result

a) We have only used two approximations: neglect of the macroscopic term in the equation of motion (7) and slow $\boldsymbol{k}$, t-dependence of $\bar{u}_m$.

b) The latter assumption can be improved systematically. The next higher order contributions are found by expanding $\bar{u}_m(\boldsymbol{p}, t')$ around t or $\boldsymbol{k}$.

The following additional terms in the transport equation arise:

α) $\partial_t\, \bar{u}_m(\boldsymbol{k}, t)$, with a transport coefficient like (11), but an additional factor t' under the integral. It measures the turbulent time scale τ_{turb} of $G(t \,|\, t')$. This correction therefore arises, if $\tau_{\text{turb}} \cdot \partial/\partial t \approx 1$, i. e. if the macroscopic time development is comparable to the turbulent time scale. $\partial_t\, \bar{u}_m$ gives rise to nonlinear, higher order transport equations by using (6 a) again.

β) $\partial_k\, \bar{u}_m(\boldsymbol{k}, t)$, with a transport coefficient, which measures $\boldsymbol{p} - \boldsymbol{k} \approx \boldsymbol{q} - \boldsymbol{q}'$, i. e. the wave number scale λ^{-1} of the turbulent motion, $G(\boldsymbol{q} \,|\, \boldsymbol{q}')$. In position space this term is *non* local. It is important, if the macroscopic scale becomes comparable with the turbulent scale.

c) The most general form of the transport contribution from turbulence leads to a fluid with memory in time and nonlocality in space. $\bar{u}_m(\boldsymbol{p}, t')$ then remains under the integral.

This shows that a simple incompressible fluid behaves like a non Newtonian, highly viscous fluid (see e. g. [17, 18]), if the equilibrium is largely disturbed. We have also derived time correlation functions for the relevant transport coefficients. They are very similar to the corresponding ones for molecular transport coefficients in non Newtonian fluids [19].

d) Let us simplify the transport coefficient (11) and reduce it to its essential features. The Green's function G is needed for the fluctuations in Eq. (11). These small eddies can be imagined as homogeneous and isotropic.

$$G_{ir}(\boldsymbol{q}\,t \,|\, \boldsymbol{q}'\,t') \approx \delta(\boldsymbol{q}, \boldsymbol{q}')\, \delta_{ir}\, G(q; t, t'). \qquad (12)$$

The same argument gives momentum conservation in the time integrated expectation value, $\langle\ \rangle \sim \delta(\boldsymbol{k}, \boldsymbol{p})$. Finally the transversal nature of $\boldsymbol{u}'$ tells us

$$\int \mathrm{d}t' \langle u_j'(\boldsymbol{k} - \boldsymbol{q}, t)\, G\, u_m'(\boldsymbol{q} - \boldsymbol{k}, t') \rangle = \int \mathrm{d}t'\, P_{jn}(\boldsymbol{q} - \boldsymbol{k}) \tfrac{1}{2} \langle u_s'(\boldsymbol{k} - \boldsymbol{q}, t) G_q\, u_s'(\boldsymbol{q} - \boldsymbol{k}, t') \rangle.$$

This gives a simple form for the eddy viscosity

$$\nu_{lm}(k, t) = \frac{1}{8\,k^2} \sum_q P_{lji}(\boldsymbol{k})\, P_{inm}(\boldsymbol{q})\, P_{jn}(\boldsymbol{k} - \boldsymbol{q}) \cdot \int^t \mathrm{d}t' \langle u_s'(\boldsymbol{k} - \boldsymbol{q}, t)\, G(\boldsymbol{q}; t, t')\, u_s'(\boldsymbol{q} - \boldsymbol{k}, t') \rangle.$$

As $k_l\, \nu_{lm}(k) = 0$ and $P_{l'l}(\boldsymbol{k})\, \nu_{lm}(k) = \nu_{l'm}(k)$ we write

$$\nu_{lm}(k) = P_{lm}(\boldsymbol{k})\, \nu_{\text{turb}}(k). \qquad (13)$$

[17] C. Truesdell and W. Noll, Nonlinear Field Theories of Mechanics, Handbuch d. Phys. III/3, Springer-Verlag, Berlin 1965.

[18] B. D. Coleman, H. Markovitz, and W. Noll, Viscometric Flows of Non-Newtonian Fluids (Springer Tracts in Nat. Philos., Vol. 5) 1966.

[19] S. Grossmann, Z. Phys. **233**, 74 [1970].

With the abbreviation

$$u_s'(\boldsymbol{q}-\boldsymbol{k},\tau;\boldsymbol{q};t) \equiv G(\boldsymbol{q};t,t')\, u_s'(\boldsymbol{q}-\boldsymbol{k},t'), \quad \tau \equiv t-t'$$

the turbulent viscosity appears in the very simple form $(k-q \gtrless q)$:

$$\nu_{\text{turb}}(k,t) = \sum_q a(\boldsymbol{k},\boldsymbol{q}) \int_0^\infty \mathrm{d}\tau \langle \tfrac{1}{2} u_i'(\boldsymbol{q},t)\, u_i'(\boldsymbol{q},\tau;\boldsymbol{k}-\boldsymbol{q};t)\rangle. \tag{14}$$

$a(\boldsymbol{k},\boldsymbol{q})$ is a dimensionless number of order unity,

$$a(\boldsymbol{k},\boldsymbol{q}) = \frac{1}{8k^2} P_{lji}(\boldsymbol{k})\, P_{jn}(\boldsymbol{q})\, P_{inl}(\boldsymbol{k}-\boldsymbol{q}) = a(-\boldsymbol{k},-\boldsymbol{q}). \tag{15}$$

$u_i'(\boldsymbol{q},\tau;\boldsymbol{k}-\boldsymbol{q};t)$ is the velocity component which is determined by Lagrangean time development from time t to time $t-\tau$, starting with wave number q. The k, t dependence of $\langle\ldots\rangle$ is expected to be small, if the macroscopic ensemble is not too far from equilibrium.

In general, the eddy transport coefficient $\nu_{lm}(k)$ depends on k. This gives a nonlocal position space mean velocity equation. Scattering experiments would measure this k-dependent quantity in the same way as a k-dependent dielectricity $\varepsilon(k)$ etc. If the macroscopic length scale is larger than the typical eddy scale, the $k\to 0$ limit is allowed to be used in Eq. (14). This is the region where turbulence only renormalizes the molecular transport coefficients.

5. The Equation for the Velocity Spectral Function

We now consider the velocity spectral function

$$U(k,t) \equiv \frac{(2\pi)^3}{V}\cdot\frac{1}{2\pi k^2} F(k,t) \tag{16}$$
$$= \langle u_l(\boldsymbol{k},t)\, u_l(-\boldsymbol{k},t)\rangle = \bar{u}_l(\boldsymbol{k},t)\,\bar{u}_l(-\boldsymbol{k},t) + \langle u_l'(\boldsymbol{k})\, u_l'(-\boldsymbol{k})\rangle.$$

We are interested in its time development in the case of homogeneous, isotropic and stationary turbulence. Nevertheless, as we must induce turbulent motion, the small k region depends on t, while the turbulent, random region is nearly stationary. Therefore the time dependent part and the stationary part have been separated. The time dependence is governed by Eq. (6 a), in which we neglect the macroscopic nonlinear term $\sim \bar{u}\,\bar{u}$.

$$\partial_t U(k,t) = \partial_t \bar{u}_l(\boldsymbol{k},t)\,\bar{u}_l(-\boldsymbol{k},t) = -2\nu k^2 U(k,t) - \tfrac{1}{4}\sum_{q q' p} \{P_{lij}(\boldsymbol{k})\, P_{rmn}(\boldsymbol{q}')$$
$$\cdot \int \mathrm{d}t'\, \bar{u}_m(\boldsymbol{p},t')\,\bar{u}_l(-\boldsymbol{k},t)\, \langle u_j'(\boldsymbol{k}-\boldsymbol{q},t)$$
$$\cdot\, G_{ir}(\boldsymbol{q}\,t\,|\,\boldsymbol{q}'\,t')\, u_n'(\boldsymbol{q}'-\boldsymbol{p},t')\rangle + (\boldsymbol{k} \gtrless -\boldsymbol{k})\}.$$

We use some approximations established in Section 4 to simplify the expectation value, namely Eq. (12) together with $\boldsymbol{p}\approx\boldsymbol{k}$. Moreover, introduce the two-time correlation function [2, 3]

$$U(k;t,t') = \langle u_l(\boldsymbol{k},t)\, u_l(-\boldsymbol{k},t')\rangle \underset{(k\ \text{small})}{\approx} \bar{u}_l(\boldsymbol{k},t)\,\bar{u}_l(-\boldsymbol{k},t').$$

The last equality holds for small k, as then the fluctuations are small, see Figure 3. This gives

$$\bar{u}_m(\boldsymbol{p},t')\,\bar{u}_l(-\boldsymbol{k},t) \approx \tfrac{1}{2} P_{ml}(\boldsymbol{k})\, U(k;t',t). \tag{17}$$

The tensor properties of the expectation value contribute an additional factor $\tfrac{1}{2}P_{jn}(\boldsymbol{k}-\boldsymbol{q})$. With $P_{ml}(\boldsymbol{k})\,P_{lij}(\boldsymbol{k}) = P_{mij}(\boldsymbol{k})$ and $k-q \gtrless q$ we find

$$(\partial_t + 2\nu k^2)\, U(k,t) = -\tfrac{1}{2}k^2 \sum_q a(\boldsymbol{k},\boldsymbol{q}) \int \mathrm{d}t'\, U(k;t'\,t) \tag{18}$$
$$\{\langle u_i'(\boldsymbol{q},t)\, G(\boldsymbol{k}-\boldsymbol{q}\,|\,t,t')\, u_i'(\boldsymbol{q},t')\rangle + (\boldsymbol{q} \gtrless -\boldsymbol{q})\}.$$

The nonlinear Green's function G depends on u. Therefore we can introduce some average, $\bar{G}$. If we take $\bar{G}$ out of $\langle\ldots\rangle$ this latter is $U(q;t,t')$.

The remaining term $\langle u'\,G'\,u'\rangle$ is of higher order and may therefore be neglected. Then we have derived one of Kraichnan's DIA equations:

$$\partial_t U(k,t) + 2\nu k^2 U(k,t) = -\tfrac{1}{2}k^2 \sum_q a(\boldsymbol{k},\boldsymbol{q}) \int_{t_0}^{t} \mathrm{d}t'$$
$$\cdot \{U(k\,|\,t,t')\, \bar{G}(\boldsymbol{k}-\boldsymbol{q}\,|\,t,t')\, U(q\,|\,t,t') + (\boldsymbol{q} \gtrless -\boldsymbol{q})\}. \tag{19}$$

$\bar{G}(q;t,t')$ is the average Lagrangean type Green's function.

The same method evidently can be used to derive the equation for the two time correlation function $\partial_t U(k;t,t')$. An equation for the Green's function $G(q\,|\,t,t')$ may be derived within this frame too. We only need the idea, as the result again is one of Kraichnan's equations.

It is $\partial_t G = -G(\partial_t G^{-1})\, G$. The only explicit t-dependence in the matrix G^{-1} is via u_j. For $\partial_t u$ one has to use the equation of motion (3 b). Finally the mean value is taken. The term $\sim \nu u \approx 0$ in stationary turbulence, the quadratic term is $\sim U$. Higher order correlations have to be neglected, which is usual in deriving Kraichnan type equations, see also PAO [20]. We are left with the equation $\partial_t G = -\sum_q \int \mathrm{d}t\, G\, U\, G$.

We summarize: The method of separating the macroscopic equation of motion from the equation for the fluctuations, and the formal integration of the later one, Eq. (8), not only gives the transport equation with time correlation function representation of the turbulent coefficients, it also reduces to Kraichnan type equations, if the same assumptions are used (which, by the way, are also made by Kraichnan). This establishes the connection between both types of turbulent theories.

As the transport form of turbulence motion is very useful in applications we consider now the relaxation approximation of the transport coefficients and finally study the temperature field, coupled to the velocity.

6. Relaxation Approximation for the Eddy Viscosity

As a simple approximation we assume an exponential decay of the correlation $G(q;t,t')$ with decay constant γ. Then the expression (14) can be simplified.

$$\nu_{\text{turb}} = \sum_q \frac{a(\boldsymbol{k},\boldsymbol{q})}{\gamma(\boldsymbol{k}-\boldsymbol{q})} \langle \tfrac{1}{2} u_i'(\boldsymbol{q})\, u_i'(-\boldsymbol{q}) \rangle = \int \frac{\mathrm{d}\Omega_q}{4\pi} \int \mathrm{d}q \frac{a(\boldsymbol{k},\boldsymbol{q})}{\gamma(\boldsymbol{k}-\boldsymbol{q})} F'(q). \quad (20)$$

$F'(q)$ is the spectral function for the turbulent region. At small q it vanishes, as there are only a few fluctuations. The integral therefore starts at k_b, the boundary of the fluctuations.

Equation (20) expresses the turbulent viscosity coefficient by the eddy probability $\sim F'(q)$ and its decay constant $\gamma(q)$. I'll give information about it first by simple dimensional arguments, then by more rigorous calculation of the Green's function.

a) Eddy Decay Constant from Dimensional Analysis

For very short wave length, i. e. very large q, the decay constant $\gamma(q)$ will be determined by molecular viscosity, i. e. $\gamma(q) \approx \nu q^2$. In the inertial region we write $\gamma(q) \equiv q\, u(q)$. The quantity $u(q)$ is a typical velocity change in the Lagrangean moved fluid element. From dimensional arguments (LANDAU–LIFSCHITZ [10]) in the time τ_q this must be $u(q) \sim [\varepsilon_{\text{dis}}\, \tau_q]^{1/2}$ (ε_{dis} is the rate of energy dissipation). Thus

$$\gamma(q) \sim q\, [\varepsilon_{\text{dis}}\, \gamma(q)^{-1}]^{1/2}, \quad \text{giving}$$
$$\gamma(q) \sim \varepsilon_{\text{dis}}^{1/3}\, q^{2/3}, \quad \text{or} \quad u(q) \sim \varepsilon_{\text{dis}}^{1/3}\, q^{-1/3}. \quad (21)$$

Remark: A simple guess might be $u(q) \approx$ const, thus $\gamma(q) \sim q$. Making the same analysis as with (21), one would get $F(k) \sim k^{-2/3}$ instead of the Kolmogoroff law.

The ansatz (21) closes the hierarchy, because now ν_{turb} is uniquely determined by the spectrum $F(q)$. Combining the constant of proportionality with the angular average of $a(\boldsymbol{k},\boldsymbol{q})$ to c_0, we have

$$\nu_{\text{turb}}(k;k_b) = c_0\, \varepsilon_{\text{dis}}^{1/3} \int_{k_b}^{\infty} \mathrm{d}q \frac{F(q)}{|\boldsymbol{k}-\boldsymbol{q}|^{2/3}}, \quad \text{inertial range.} \quad (22)$$

This formula has a similar interpretation as e. g. HEISENBERG's [8], but it is different in its analytic form.

b) Decay Constant and Eddy Viscosity from Perturbation Theory

Perturbation theory in turbulence has been introduced first by WYLD [4]. Recently his method has been applied to dynamical phenomena near critical points (for a survey see [21]). Instead of earlier theories the causal Green's function is used, defined for isotropic, homogeneous turbulence by

$$\langle u_i(\boldsymbol{k},t)\, u_j(-\boldsymbol{k}) \rangle = G_k(t) \langle u_i(\boldsymbol{k})\, u_j(-\boldsymbol{k}) \rangle = \tfrac{1}{2} U_k\, G_k(t)\, P_{ij}(\boldsymbol{k}), \quad t \geqq 0. \quad (23)$$

Using the same technique as Wyld, Kawasaki and others one gets the following nonlinear equation for the Fourier transform of $G_k(t)$:

$$G_k^{-1}(\omega) = -i\,\omega + k^2 \nu + k^2 \sum_q \cdot a(\boldsymbol{k},\boldsymbol{q})\, U_q \int_{-\infty}^{+\infty} \frac{\mathrm{d}\omega'}{\pi} G_{k-q}(\omega')\, G_q(\omega-\omega'). \quad (24)$$

The only question is the value of the effective interaction $a(\boldsymbol{k},\boldsymbol{q})$. It can be shown to be the *same*, which we found in Eq. (14). Of course, there are assumptions involved to derive Eq. (24):

[20] YIH-HO PAO, Boeing Scientific Res. Lab. No. D 1-82-0836, 1969.

[21] K. KAWASAKI, Ann. Phys. (New York) **61**, 1 [1970].

stochastic forces, which make the Green's function $\mathring{G}_k^{-1}(\omega) = -i\,\omega + k^2\,\nu$ stationary in time; Gaussian distribution of the $u_i(\boldsymbol{k})$ at initial time, which allows to factorize higher order correlations; repeated use of isotropy and momentum conservation (homogeneity).

If the selfenergy ($\sim G^2$) does not depend too much on ω, the relaxation approximation holds, $G_k(t) = e^{-\Gamma_k t}$.

$$\frac{1}{k^2}\,\Gamma_k = \nu + 2\sum_q \frac{a(\boldsymbol{k},\boldsymbol{q})\;U_q}{\Gamma_{k-q}+\Gamma_q}. \qquad (25)$$

This equation determines the decay constant Γ_k as functional of the turbulent "interaction" matrix element $a(\boldsymbol{k},\boldsymbol{q})$ *and* the turbulent spectrum F_q. By scaling one gets [with Eq. (21)] in the inertial region

$$F_q \sim \Gamma_k^2/k^3 \sim k^{-5/3}. \qquad (26)$$

If $\Gamma_q \approx \Gamma_{k-q}$ and the Heisenberg cascade idea is used, that only $q \gtrsim k$ are needed to find the damping of the mode k, the solution of (25) reads

$$\frac{1}{k^2}\,\Gamma_k = \left(\nu^2 + 4\,\bar{a}\int_k^\infty \mathrm{d}q\;\frac{F(q)}{q^2}\right)^{1/2}. \qquad (27)$$

$\bar{a}$ is the average value of the turbulent interaction, Eq. (15). As Eq. (24) shows, the selfenergy is k^2 times the turbulent viscosity. Thus

$$\nu_{\mathrm{turb}}(k) := \frac{\Gamma_k}{k^2} - \nu = \sqrt{\nu^2 + 4\,\bar{a}\int_k^\infty \mathrm{d}q\;\frac{F(q)}{q^2}} - \nu$$

$$\approx \sqrt{4\,\bar{a}\int_k^\infty \mathrm{d}q\;\frac{F(q)}{q^2}}\,. \qquad (28)$$

In the viscous range, $k \to \infty$, $\nu_{\mathrm{turb}}(k) \to 0$; in the inertial subrange it differs from the simple dimensional form (22) as well as from Heisenberg's ansatz $\sim \int_k^\infty \sqrt{[F(q)/q^3]}\cdot \mathrm{d}q$.

Now the turbulent spectral function can be determined in relaxation approximation.

7. The Turbulent Spectrum

In order to determine F as function of k_b, the boundary between energy containing macroscopic modes and energy transfering turbulent modes we go back to Eq. (18) in which the r.h.s. now reads $-2\,\nu_{\mathrm{turb}}\,k^2\,U(k,t)$.

$$\partial_t F(k,t) + 2(\nu+\nu_{\mathrm{turb}})\,k^2\,F(k,t) = 0. \qquad (29)$$

This is integrated over the energy containing range $0\ldots k_\mathrm{b}$. As $\partial_t \int_0^{k_\mathrm{b}} F(p,t)\,\mathrm{d}p = -\varepsilon_{\mathrm{dis}}$, one gets the Heisenberg equation

$$\varepsilon_{\mathrm{dis}} = 2(\nu+\nu_{\mathrm{turb}})\int_0^{k_\mathrm{b}} p^2\,F(p)\,\mathrm{d}p. \qquad (30)$$

Its solution in the inertial subrange ($\nu \ll \nu_{\mathrm{turb}}$) with the dimensional form of the eddy viscosity, Eq. (22), is

$$F(k_\mathrm{b}) = \sqrt{\frac{8}{9\,c_0}}\;\varepsilon_{\mathrm{dis}}^{2/3}\,k_\mathrm{b}^{-5/3}. \qquad (31)$$

Using $\gamma \sim q$ would have given $F \sim k^{-3/2}$ in relaxation approximation, which has been found by Kraichnan in non-Lagrangean DIA.

If the eddy viscosity from perturbation theory is used, Eq. (28), also the Kolmogoroff law $\sim k^{-5/3}$ is found, but in addition the absolute factor can be calculated.

$$F(k_\mathrm{b}) = C\,\varepsilon_{\mathrm{dis}}^{2/3}\,k_\mathrm{b}^{-5/3}, \qquad C = \frac{2}{3(\bar{a})^{1/3}}. \qquad (32)$$

The angular average of $a(\boldsymbol{k},\boldsymbol{q})$ is

$$4\,\overline{a(k,\mathrm{q})} = \int \frac{\mathrm{d}\Omega_q}{4\pi}\,k_j^0\,P_{jn}(q^0)\,k_n^0 \cdot [1 - k_i^0\,P_{il}(k-q)\,k_l^0],$$

$$4\,\bar{a} = \left\langle (1-\zeta^2)\,\frac{1-2x\zeta+x^2\zeta^2}{1-2x\zeta+x^2}\right\rangle_\zeta, \qquad (33)$$

$$x \equiv q/k, \quad \zeta \equiv \sphericalangle\,\boldsymbol{k},\boldsymbol{q}.$$

Especially $q \approx k$ is needed. If $x = 1$ or 2 then $8\,\bar{a} = 2/3$ or $1/2$, giving $C = 1.53$ or 1.67. The experimental value is reported as 1.5 by Leith; Kraichnan's LHDIA gives $C = 1.77$.

Equation (30) together with Eq. (28) can be solved exactly. With the abbreviation

$$\frac{2\,\nu}{\varepsilon_{\mathrm{dis}}}\int_0^k p^2\,F(p)\,\mathrm{d}p =: f(k),$$

$$\frac{1}{k^2}\,f'(k)\,\frac{\varepsilon_{\mathrm{dis}}}{2\,\nu} = F(k),$$

f monotonously increasing from 0 to 1, it reads

$$\frac{1}{f} - 1 = \sqrt{\frac{2\,\bar{a}\,\varepsilon_{\mathrm{dis}}}{\nu^3}\int_k^\infty \mathrm{d}q\;\frac{f'(q)}{q^4}}\,.$$

Taking the square and the k-derivative one gets

$$\frac{f^3}{1-f} = \frac{\nu^3}{\bar{a}\,\varepsilon_{\text{dis}}}\,k^4. \tag{34}$$

The small k region ($f \sim k^{4/3}$, thus $F \sim k^{-5/3}$) is the inertial subrange; if $k \to \infty$, i. e. in the viscous range, $f \nearrow 1$. The boundary between both regions evidently is given by $(k/k_\nu)^4 \equiv x^4 = 1$,

$$k_\nu := \left(\frac{\bar{a}\,\varepsilon_{\text{dis}}}{\nu^3}\right)^{1/4}. \tag{35}$$

The solution for small x has already been given in Eq. (32). For large x one finds $f = 1 - x^{-4}$, thus $F \sim k^{-7}$. The factor is 3-times that of Heisenberg's [8] result:

$$F(k) = \begin{cases} \dfrac{2}{3(\bar{a})^{1/3}}\,\varepsilon_{\text{dis}}^{2/3}\,k^{-5/3} \\ \dfrac{2\,\bar{a}\,\varepsilon_{\text{dis}}^2}{\nu^4}\,k^{-7} \end{cases}$$

$$= \frac{2}{3(\bar{a})^{3/4}}\,(\varepsilon_{\text{dis}}\,\nu^5)^{1/4} \begin{cases} (k/k_\nu)^{-5/3} & k < k_\nu\,, \\ 3\,(k/k_\nu)^{-7} & k > k_\nu\,. \end{cases} \tag{36}$$

Finally we can describe the frequency spectrum of turbulent motion. $G_k(\omega)$ is a Lorentz distribution with center $\omega = 0$ and half width Γ_k. Using Eq. (27) and the spectrum (36) one gets for the decay constant relative to the molecular half width

$$\frac{\Gamma_k}{k^2\,\nu} = \begin{cases} (k/k_\nu)^{-4/3} & \text{inertial subrange, } k < k_\nu\,, \\ 1 + \tfrac{1}{2}\,(k/k_\nu)^{-8} & \text{viscous region, } \quad k > k_\nu\,. \end{cases} \tag{37}$$

It is this spectrum, which is measured in time correlation experiments, as

$$\varphi(\omega; x) = \int_{-\infty}^{+\infty} \mathrm{d}t \langle u_i(\boldsymbol{x}, t)\, u_i(0,0)\rangle\, e^{i\omega t}$$
$$= \int \mathrm{d}k\, F_k\, \frac{\sin k x}{k x}\, \frac{4\,\Gamma_k}{\Gamma_k^2 + \omega^2}\,. \tag{38}$$

The ω behaviour of φ can be connected with the k behaviour of F_k and Γ_k by scaling arguments. Introduce the "critical" indices of damping ($1/a$) and spectrum (β) by

$$\Gamma(\lambda^a k) = \lambda\,\Gamma(k)\,; \quad F(\lambda k) = \lambda^{-\beta}\,F(k)\,.$$

Then $\varphi \sim \omega^{a - a\beta - 1}$. It is $a = 3/2$, $\beta = 5/3$ in the inertial range and $a = 1/2$, $\beta = 7$ in the viscous subrange. Therefore ($x \approx 0$)

$$\varphi(\omega) \sim \begin{cases} \omega^{-2} & \text{inertial range,} \\ \omega^{-4} & \text{viscous range.} \end{cases} \tag{39}$$

Especially the ω^{-2} behaviour reflects the Kolmogoroff spectrum $\sim k^{-5/3}$ in k-space. If $F_k \sim k^{-3/2}$ and $\Gamma_k \sim k$, one has $\varphi \sim \omega^{-3/2}$.

8. Turbulent Heat Transport

The effect of turbulence on pressure and temperature is described by Eq.'s (6 b), (6 c). The turbulent contribution to the pressure is given by the same time correlation function of the velocity fluctuations, which is needed for the eddy viscosity, namely (9). In the same approximation as in Section 4 we get

$$\bar{P}_{\text{turb}}(\boldsymbol{k}, t) = \bar{u}_m(\boldsymbol{k}, t) \sum_q b_m(\boldsymbol{k}, \boldsymbol{q}) \int_0^\infty \mathrm{d}\tau \langle \tfrac{1}{2}\, u_i'(\boldsymbol{q}, t)$$
$$\cdot\, u_i'(\boldsymbol{q}, \tau; \boldsymbol{k}-\boldsymbol{q}; t)\rangle \tag{40}$$

with $$b_m(\boldsymbol{k}, \boldsymbol{q}) = \frac{i}{2}\, k_l^0\, k_j^0\, P_{jn}(\boldsymbol{q})\, P_{lmn}(\boldsymbol{k}-\boldsymbol{q})\,. \tag{41}$$

As $b_m \sim k$, this turbulent contribution is connected with mean velocity inhomogeneities, $\bar{u}_m k_n$.

The temperature Eq. (6 b) contains a new type of correlation, $\langle u'\, T'\rangle$. But as the turbulent effects originate in the nonlinearity of the velocity equation it is to be expected that this correlation can also be reduced to the eddy viscosity. To show this we consider the equation for the temperature fluctuation, which is induced by the velocity fluctuations via Equation (3 c).

$$\partial_t T'(\boldsymbol{k}) + i\,k_j \sum_q \{u_j'(\boldsymbol{k}-\boldsymbol{q})\,\bar{T}(\boldsymbol{q})$$
$$+ \bar{u}_j(\boldsymbol{k}-\boldsymbol{q})\,T'(\boldsymbol{q}) + u_j'(\boldsymbol{k}-\boldsymbol{q})\,T'(\boldsymbol{q}) \tag{42}$$
$$- \langle u_j'(\boldsymbol{k}-\boldsymbol{q})\,T'(\boldsymbol{q})\rangle\} = -\chi\,k^2\,T'(\boldsymbol{k})\,.$$

Again neglecting the slowly varying correlation $\langle \ldots \rangle$ we get the analogon to Eq. (7).

$$\sum_q \left[(\partial_t + \chi k^2)\,\delta(\boldsymbol{k}, \boldsymbol{q}) + i\,k_j\,u_j(\boldsymbol{k}-\boldsymbol{q}, t)\right]\, T'(\boldsymbol{q}, t)$$
$$= -i\,k_i \sum_p \bar{T}(\boldsymbol{p}, t)\, u_i'(\boldsymbol{k}-\boldsymbol{p}, t)\,. \tag{43}$$

In contrast to Eq. (7) here the r.h.s. is indeed a known inhomogeneity and the differential operator [...] on the l.h.s. is also given, at least in principle, as the velocity u_j and its fluctuation could be calculated without knowing T'.

Introducing the Green's function $\tilde{G}$ of Eq. (43), we get the temperature fluctuation in terms of the velocity field.

$$T'(\boldsymbol{q},t) = -i\int_{t_0}^{t} \mathrm{d}t' \sum_{q',p} \tilde{G}(\boldsymbol{q}\,t\,|\,\boldsymbol{q}'\,t')\; q_i'\,\bar{T}(\boldsymbol{p},t') \cdot u_i'(\boldsymbol{q}'-\boldsymbol{p},t'). \quad (44)$$

This is used in the turbulent correlation function of Equation (6 b).

$$-i\,k_j \sum_q \langle u_j'(\boldsymbol{k}-\boldsymbol{q})\; T'(\boldsymbol{q})\rangle = -\sum_{q,q',p}\int_{t_0}^{t} \mathrm{d}t'\, k_j\, q_i'\,\bar{T}(\boldsymbol{p},t') \quad (45)$$
$$\langle u_j'(\boldsymbol{k}-\boldsymbol{q},t)\;\tilde{G}(\boldsymbol{q}\,t\,|\,\boldsymbol{q}'\,t')\; u_i'(\boldsymbol{q}'-\boldsymbol{p},t')\rangle.$$

The factor in front of the correlation function is a mean temperature gradient. This suggests to introduce an eddy heat conductivity $\chi_{\text{turb}}(k)$, which describes the turbulent effects on heat conduction (to lowest order).

$$-i\,k_j \sum_q \langle u_j'(\boldsymbol{k}-\boldsymbol{q})\; T'(\boldsymbol{q})\rangle = -k^2\,\chi_{\text{turb}}(k)\;\bar{T}(\boldsymbol{k},t) \quad (46)$$

with the transport coefficient

$$\chi_{\text{turb}}(k,t) = \sum_{q,q'}\int_{t_0}^{t} \mathrm{d}t'\,\frac{k_j\,q_i'}{k^2} \cdot \langle u_j'(\boldsymbol{k}-\boldsymbol{q},t)\;\tilde{G}(\boldsymbol{q}\,\mathrm{t}\,|\,\boldsymbol{q}'\,t')\; u_i'(\boldsymbol{q}'-\boldsymbol{k},t')\rangle. \quad (47)$$

Use $q' \approx q$ for the fluctuations, separate the transversal projector, and transform $k-q \rightleftarrows q$.

$$\chi_{\text{turb}}(k,t) = \sum_q c(\boldsymbol{k},\boldsymbol{q}) \int_0^\infty \mathrm{d}\tau \langle \tfrac{1}{2}\, u_i'(\boldsymbol{q},t)\; \tilde{G}(\boldsymbol{k}-\boldsymbol{q},\tau) \cdot u_i'(\boldsymbol{q},t-\tau)\rangle \quad (48)$$

with

$$c(\boldsymbol{k},\boldsymbol{q}) = k_i^0\,k_j^0\,P_{ij}(\boldsymbol{q}) = 1-(\boldsymbol{k}^0\cdot\boldsymbol{q}^0)^2;\; \bar{c} = 2/3. \quad (49)$$

The difference of the coefficients $a(\boldsymbol{k},\boldsymbol{q})$ in Eq. (18) and $c(\boldsymbol{k}^0\cdot\boldsymbol{q}^0)$ as well as that of G and $\tilde{G}$ may be not too important; then

$$\chi_{\text{turb}} \approx \nu_{\text{turb}}\,. \quad (50)$$

As the velocity eddies determine the turbulent heat transport, also the spectral function of the temperature field is to be expected as fixed by the velocity spectrum, $F(k)$.

The same method is used as in Sect. 6. Define the temperature spectral function by

$$R(k,t) = \langle T(\boldsymbol{k},t)\; T(-\boldsymbol{k},\mathrm{t})\rangle = \bar{T}(\boldsymbol{k},t)\;\bar{T}(-\boldsymbol{k},t) + \langle T'(\boldsymbol{k})\; T'(-\boldsymbol{k})\rangle.$$

Then approximately

$$\partial_t R(k,t) = -2\,(\chi + \chi_{\text{turb}}(k))\,k^2\,R(k,t). \quad (51)$$

Integrating over the energy containing wave numbers in the spirit of the cascade technique, we get instead of Eq. (30)

$$-\partial_t \int_0^{k_b} R(p)\,\mathrm{d}p = 2\,(\chi+\chi_{\text{turb}}) \int_0^{k_b} p^2\,R(p)\,\mathrm{d}p = \text{const}\,. \quad (52)$$

This determines R as χ_{turb} contains F. If $\nu \approx \chi$ we get in the inertial range

$$R \sim k^{-5/3}\,. \quad (53)$$

In this way the spectral functions of dynamical variables, which are coupled to the turbulent velocity field, may be determined quite generally.